Tutorial class 28/3

1 Series of real numbers

Definition 1.1. *We say that* $\sum_{i=1}^{\infty} x_i$ *converge if* $\sum_{i=1}^{n} x_i$ *is a convergent sequence.*

Proposition 1.1. *(Necessary condition)* $\sum_{i=1}^{\infty} x_i$ *is convergent only if* $x_n \to 0$ *as* $n \to \infty$.

Proof. Denote $s_n = \sum_{i=1}^{n} x_i$, $L = \sum_{i=1}^{\infty} x_i$. Then $x_n = s_n - s_{n-1} \to L - L = 0$. □

Proposition 1.2. *(Cauchy Criterion)* $\sum_{i=1}^{\infty} x_i$ *is convergent if and only if* $\forall\, \epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that for all* $m, n > N$, $\sum_{i=n}^{m} x_i < \epsilon$.

Proof. Directly from the result of convergent sequence. □

2 tests for convergence

Hence we have the following comparsion test.

Corollary 2.1. *If* $\{a_k\}$, $\{b_l\}$ *are two sequence of real number in which* $0 \leq a_k \leq b_k$ *for all* $k \in]mathbbN$. *Then* $\sum a_k$ *converge if* $\sum b_k$ *converge.*

Example 2.2. *The following series are convergent.*

1. $\sum_{n=1}^{\infty} ne^{-n^2}$
2. $\sum_{n=1}^{\infty} \frac{n}{n^{2+\epsilon}-n+1}$, *where* $\epsilon > 0$.

Proof. Since $xe^{-x/2} \to 0$ as $x \to 0$, we know that $xe^{-x/2}$ is bounded by some $L > 0$ on $[0, +\infty)$. Thus

$$ne^{-n^2} \leq e^{-n^2/2} \cdot \frac{Ln}{n^2} \leq Le^{-n/2} \ \ \forall\, n \in \mathbb{N}.$$

Right hand side is clearly summable. Hence by comparsion test, the series is convergent.
Noted that

$$\frac{n}{n^{2+\epsilon} - n + 1} \leq \frac{n}{n^{2+\epsilon} - n} \leq \frac{n}{n^{2+\epsilon} - \frac{1}{2}n^{2+\epsilon}} = \frac{2}{n^{1+\epsilon}}.$$

The second inequality hold when $n \geq N(\epsilon)$. (say $N > \log_2(2+\epsilon)$) By comparsion test, $\sum_{n=N}^{\infty} \frac{n}{n^{2+\epsilon}-n+1}$ is convergent and hence the whole series converges. □

Theorem 2.3. *(Montone convergence theorem) Suppose* $x_n \geq 0$, *then* $\sum_{n=1}^{\infty} x_n$ *converge if and only if the partial sum is bounded uniformly.*

Example 2.4. *Suppose* $x_n \geq 0$ *and* $\sum x_n$ *converge. Then the following series converge.*

1. $\sum x_n^{1+\epsilon}$, *where* $\epsilon > 0$.

Proof. We have $x_n \to 0$ as $n \to \infty$. So there exists N such that for all $n > N$, $0 \le x_n \le 1$. Thus

$$0 \le x_n^{1+\epsilon} \le x_n \;\; \forall\, n > N.$$

Thus, by comparsion test or MCT, the result follows. □

2. $\sum \frac{\sqrt{x_n}}{n}$

Proof. By cauchy inequality,

$$\sum_{n=1}^{N} \frac{\sqrt{x_n}}{n} \le \left(\sum_{n=1}^{N} x_n \right)^{1/2} \left(\sum_{n=1}^{N} \frac{1}{n^2} \right)^{1/2} \le L.$$

The upper bound is due to the convergence of $\sum x_n$ and $\sum 1/n^2$. Thus the series is convergent. □

3. *Suppose $\sum a_k$ and $\sum b_k$ are two series of positive numbers such that $\lim_{k\to\infty} \frac{a_k}{b_k} = l > 0$, then $\sum a_k$ is summable if and only if $\sum b_k$ is so.*

Proof. There exists N such that for all $n \ge N$,

$$\frac{l}{2} \cdot b_n \le a_n \le 2l \cdot b_n.$$

The conclusion follows from comparison test. □

Theorem 2.5. *(Root Test)Suppose a_n is sequence of real number such that*

$$\limsup_{n\to\infty} |a_n|^{1/n} = L.$$

Then the series converges absolutely if $L < 1$, and diverge if $L > 1$.

Proof. If $0 \le L < 1$, because of the assumption, there exists $N \in \mathbb{N}$ so that

$$\sup_{k\ge n} |a_k|^{1/k} \le l = \frac{1+L}{2}, \;\; \forall n \ge N,$$

Thus, for all $n \ge N$, $|a_n| \le l^n$. But the series $b_n = l^n$ is clearly convergent. So $\sum_{n=N}^{\infty} |a_n|$ converges and hence $\sum_{n=1}^{\infty} |a_n|$.
If $L > 1$, for $l = \frac{1+L}{2}$, there exists $N \in \mathbb{N}$ such that $\sup_{k\ge n} |a_k|^{1/k} > l > 1$ for all $n > N$. So for each $n > N$, there exists a subsequence a_{n_j} so that $|a_{n_j}| \ge l^{n_j} \to +\infty$. So the series cannot be convergent. □

Example 2.6. $\sum (\frac{n}{2n+1})^n$ *is convergent.*

Proof. $|x_n|^{1/n} = \frac{n}{2n+1} \to \frac{1}{2} \in [0, 1)$. □

The existence of improper integral is similar to the convergence of series. The relationship is illustrated below.

Theorem 2.7. *Let f be positive decreasing function on $[1, +\infty)$. Then the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\int_1^{\infty} f(t)\, dt$ exists.*

Proof. Basically due to the fact that for all $k \geq 2$,

$$f(k) \leq \int_{k-1}^{k} f(t)\, dt \leq f(k-1).$$

Therefore, for all $n \geq m \geq 1$,

$$\sum_{k=m+1}^{n} f(k) \leq \int_{m}^{n} f(t)\, dt \leq \sum_{k=m}^{n-1} f(k).$$

If the integral exists, take $m = 1$ to see that partial sum is bounded and hence convergent by monotone convergent theorem.
If the series is convergent, $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$,

$$0 < \sum_{k=m}^{n} f(k) < \epsilon.$$

Thus, for all $x > y > N + 1$,

$$\int_{y}^{x} f(t)\, dt \leq \int_{[y]}^{[x]+1} f(t)\, dt \leq \sum_{k=[y]}^{[x]+1} f(k) < \epsilon.$$

So the integral exists by cauchy criterion. □

Example 2.8.

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if $p > 1$.*

   Proof. If $p \leq 0$, the series is clearly divergent by the convergent criterion. By integral test, suffices to consider the function $f(t) = \frac{1}{t^p}$ where $p > 0$. Now let us compute the integral.

   $$\int_{1}^{x} \frac{1}{t^p}\, dt = \frac{x^{1-p} - 1}{1 - p}.$$

   So the integral exists if and only if $p > 1$. □

2. *The series* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\log k)^{\alpha}}$ *converge when $\alpha > 1$.*

   Proof. The improper integral $\int_1^{\infty} f(t)\, dt$ where $f(t) = \frac{1}{t \log t^p}$ exists if $p > 1$. □

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n \log \log n}$ *diverge.*